Exercise 1

a) E is the channel's reversal potential.

When $v = E$, current stops flowing through the channel (it changes sign as v crosses E).

b)

$$x(t) = x_0(E) + \left(x_0(v_0) - x_0(E)\right)\left(1 - e^{-t/\tau}\right)$$

c) We need to find p such that

$$\sqrt[p]{I_{ion} / (v - E)}$$

Behaves as the exponential we derived in (b), converging exponentially from the initial value and the final value.
(It should be a line on a log plot).

d) Assuming x_0 attains 1 at least somewhere, it suffices to observe

$$\max_{U} \frac{I_{ion}(\infty; U)}{U - E}$$

If x_0 never attains 1, then g is not uniquely defined (I can divide x by 2, multiply g_0 by 2^p and still get the same observable values for I_{ion})

Exercise 2

$$f(h) = \begin{cases} 10 \text{ Hz} & h < 0 \\ 40 \text{ Hz} & h \geq 0 \end{cases}$$

Numerical values lookup

	$h < 0$	$h \geqslant 0$
$wf(h)$	0.5 mV	2 mV
$bf(h)$	2 mV	8 mV

$RI = 3$ mV

Until otherwise

a) We need $dh_2/dt = 0$, i.e.,

$$h_2 = RI_2 + wf(h_2) - bf(h_1)$$

$h_1 < 0$

- $h_2 < 0$: $h_2 = 3 + 0.5 - 2 = 1.5$ mV ✗
- $h_2 \geqslant 0$: $h_2 = 3 + 2 - 2 = 3$ mV ✓

$h_1 > 0$

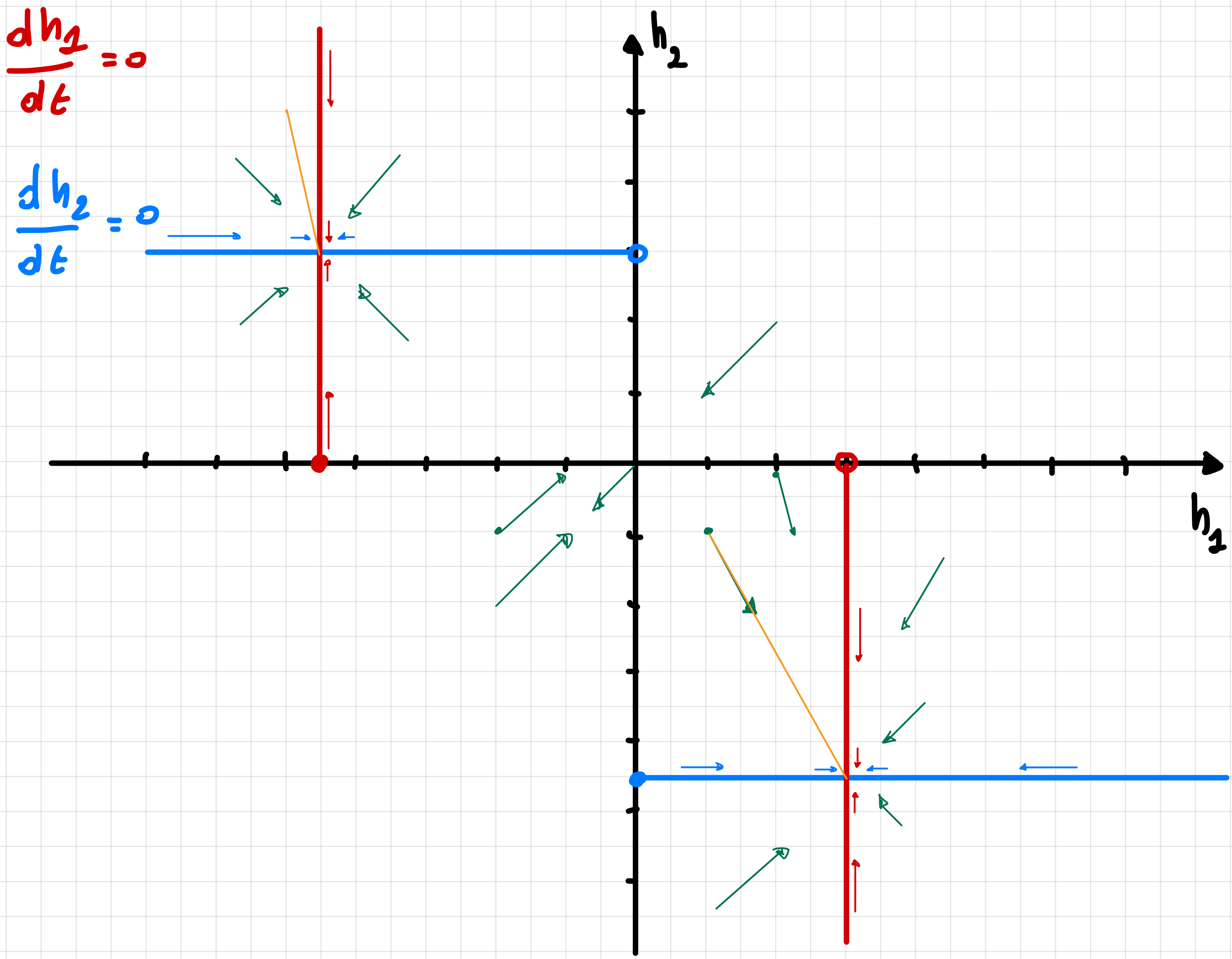
- $h_2 < 0$: $h_2 = 3 + 0.5 - 8 = -4.5$ mV ✓
- $h_2 \geqslant 0$: $h_2 = 3 + 2 - 8 = -3$ mV ✗

In summary:

$$\frac{dh_2}{dt} = 0 \iff h_2 = \begin{cases} 3\ mV & \text{iff } h_1 < 0 \\ -4.5\ mV & \text{iff } h_1 \geq 0 \end{cases}$$

b) Note that by symmetry we have

$$\frac{dh_1}{dt} = 0 \iff h_1 = \begin{cases} 3\ mV & \text{iff } h_2 < 0 \\ -4.5\ mV & \text{iff } h_2 \geq 0 \end{cases}$$



c) $(h_1, h_2) = (1, -1)$

$$\tau \frac{dh_1}{dt} = -1 + 3 + 2 - 2 = 2\,mV$$

$$\tau \frac{dh_2}{dt} = +1 + 3 + 0.5 - 8 = -3.5\,mV$$

$$(h_1, h_2) = (2, -\varepsilon)$$

$$\frac{\tau\, dh_1}{dt} = -2 + 3 + 2 - 2 = 1\,mV$$

$$\frac{\tau\, dh_2}{dt} = \varepsilon + 3 + 0.5 - 8 = -4.5\,mV$$

(see plot!)

d) $$\lim_{h_1 \to \infty} \frac{dh_1}{dt} < 0$$

To the RIGHT we go LEFT

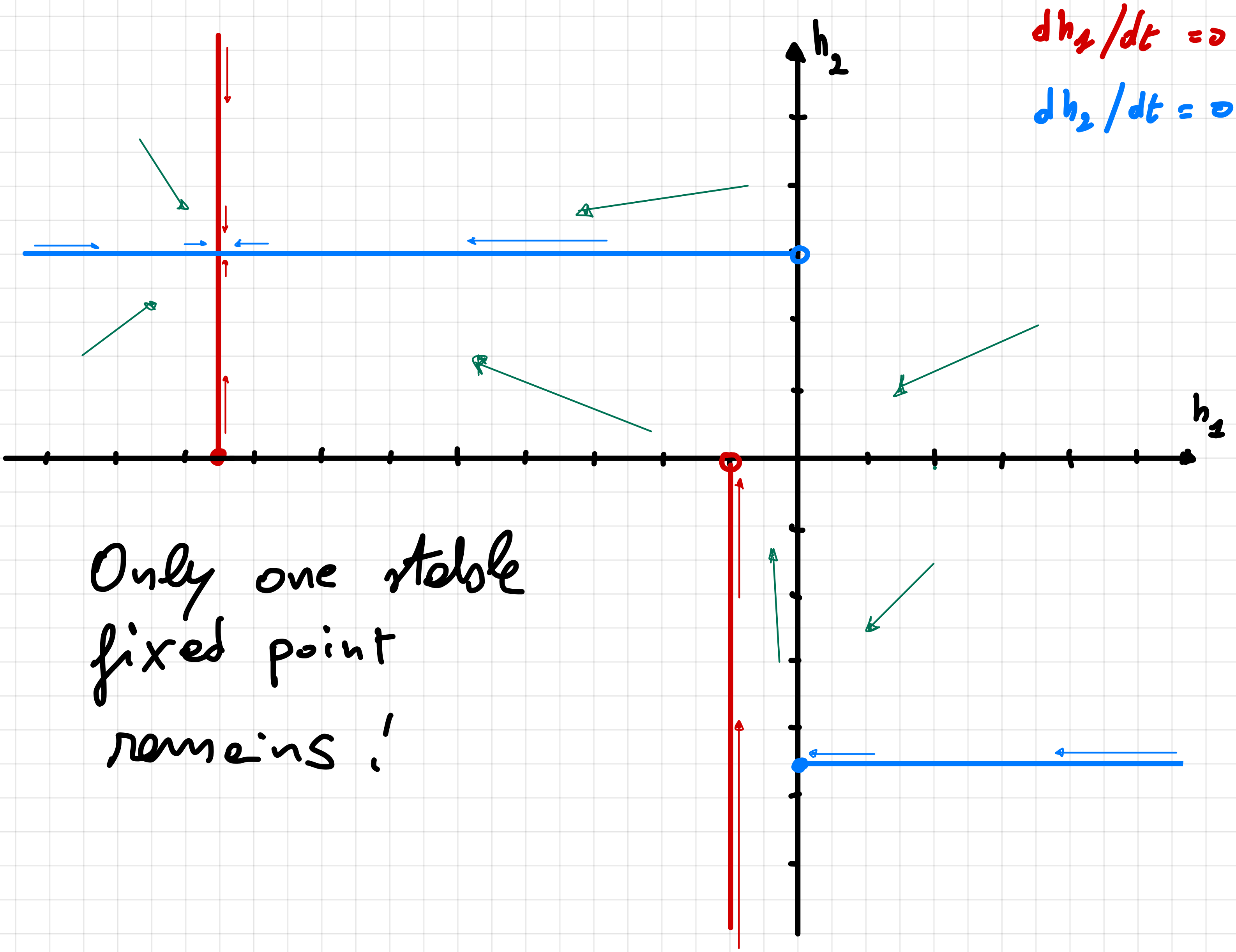
$$\lim_{h_2 \to \infty} \frac{dh_2}{dt} < 0$$

ABOVE we go DOWN

e) (See plot!)

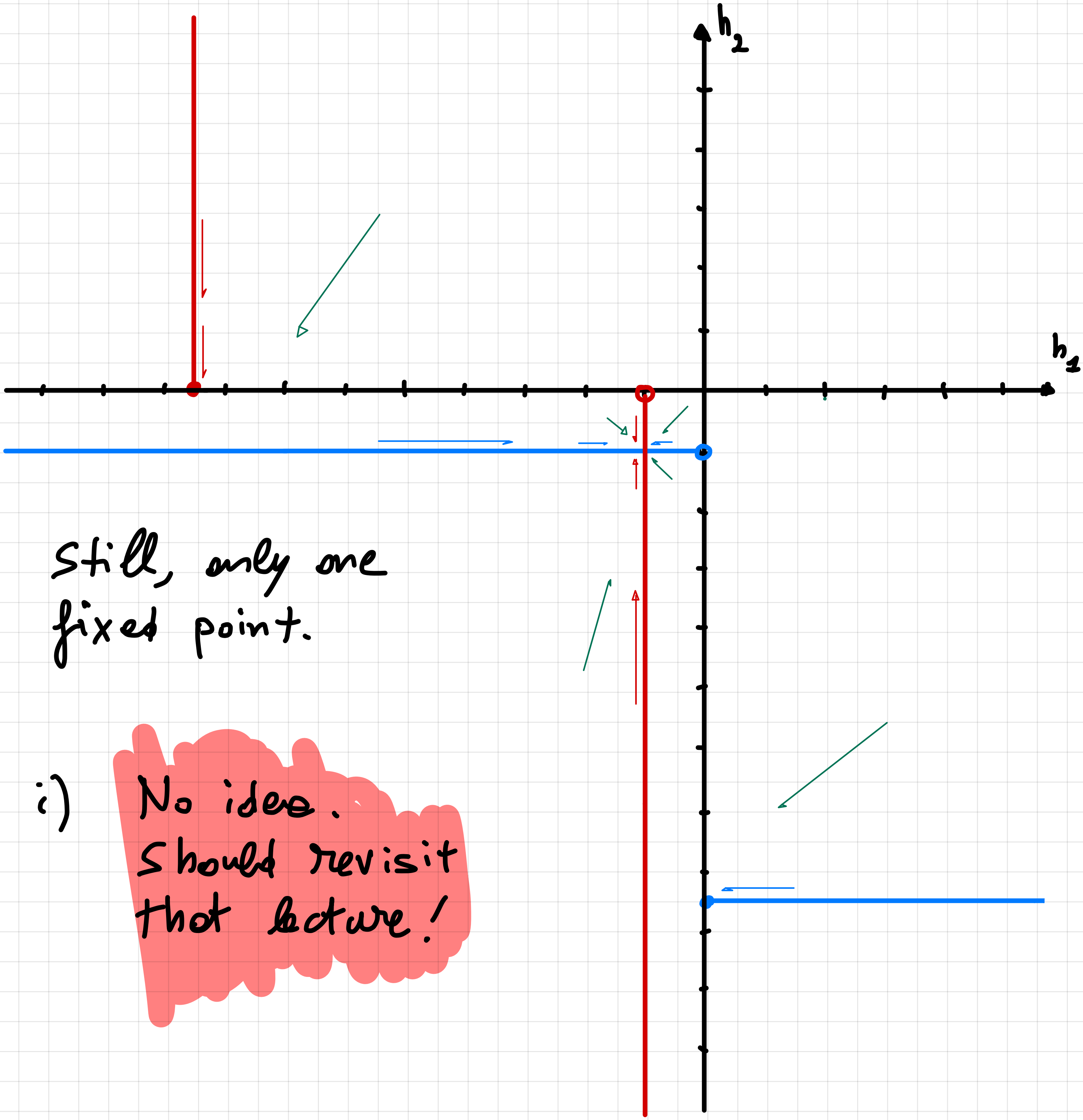
f) (See plot!)

g) The $dh_1/dt=0$ nullcline shifts left by 4 mV. The dh_2/dt nullcline stays the same.



Only one stable fixed point remains!

h) $dh_1/dt = 0$ $dh_2/dt = 0$

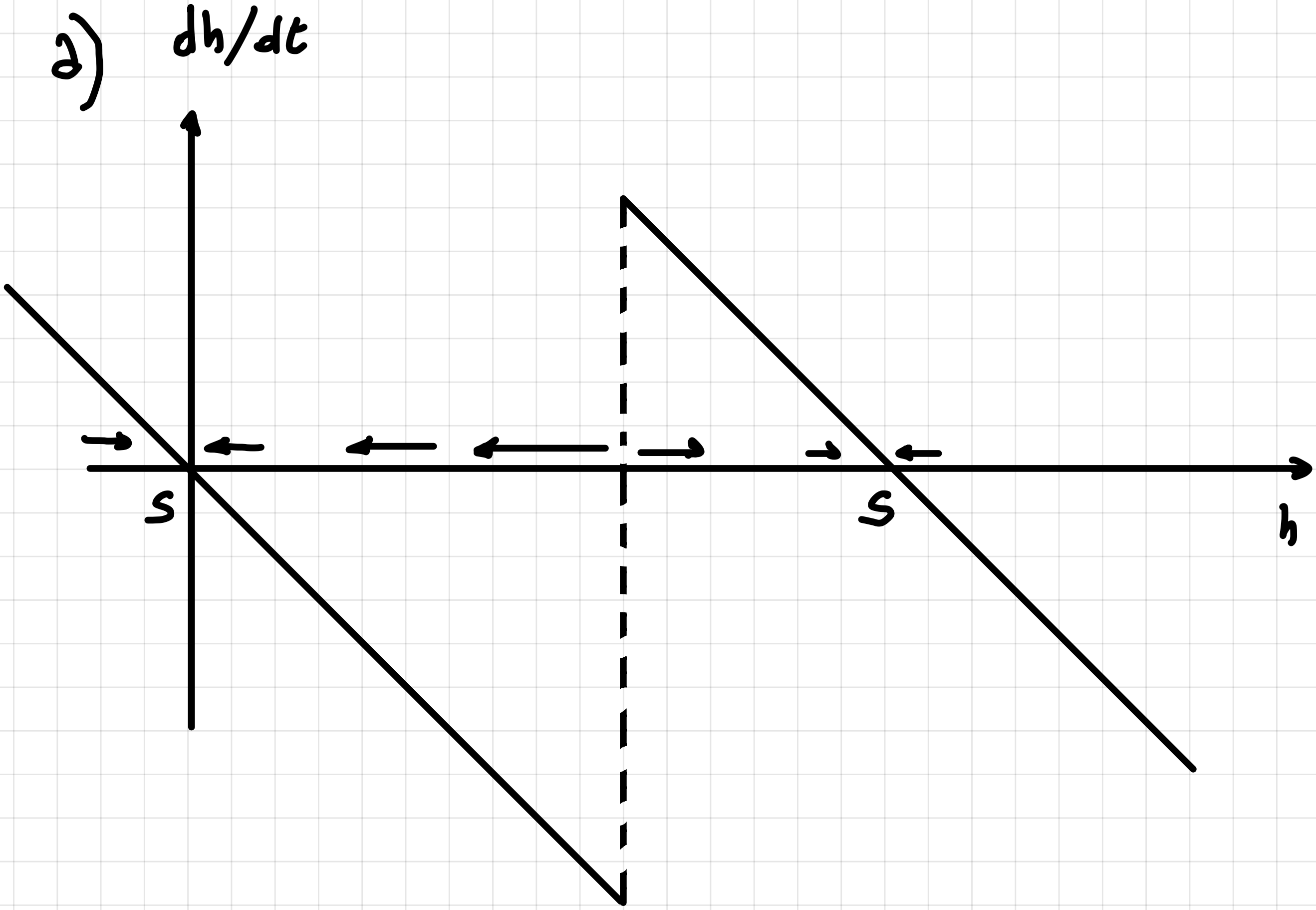


Still, only one fixed point.

i) No idea. Should revisit that lecture!

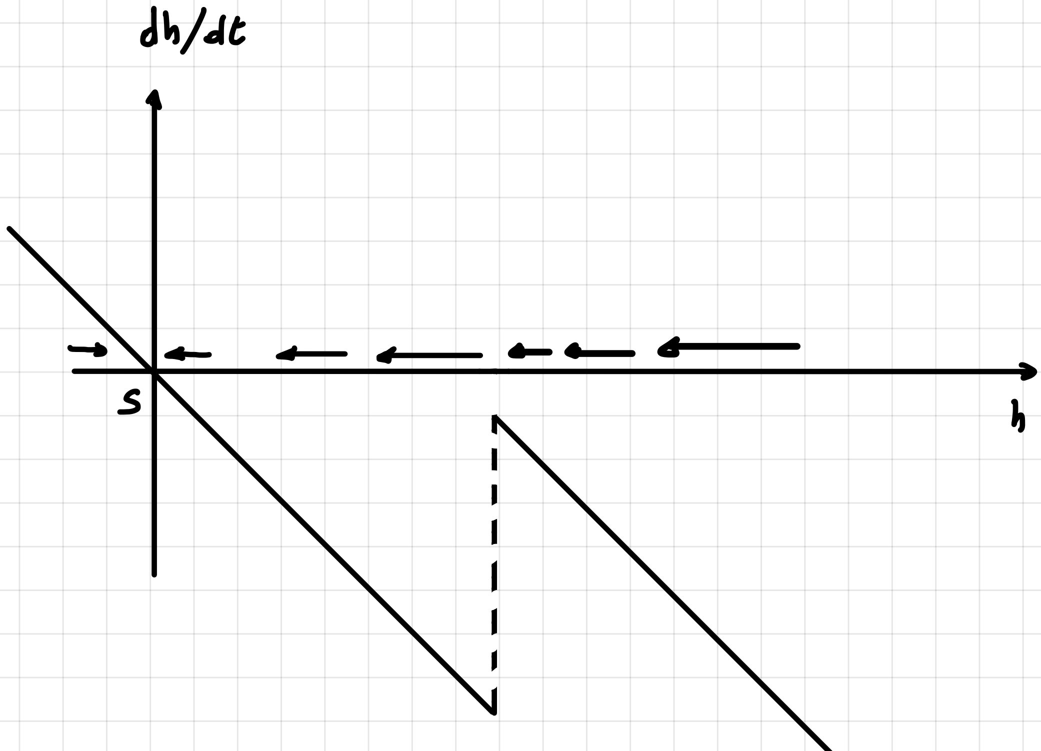
Exercise 3

a)

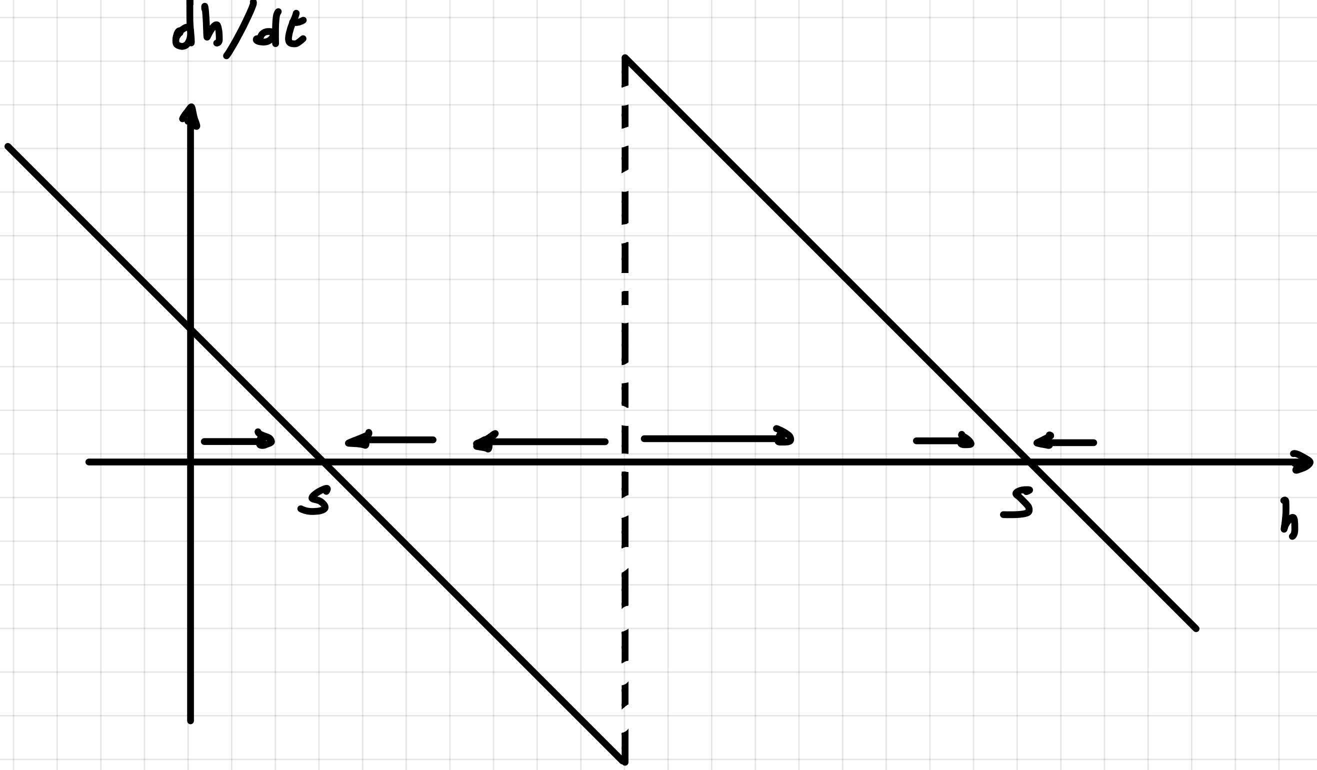


b) (See plot!)

c)



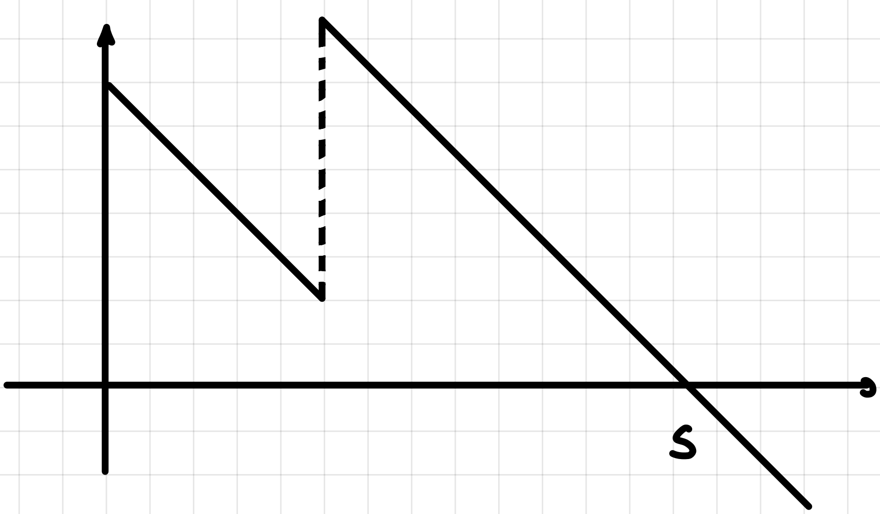
d)



e)

$$\theta + R I_0 > 0$$

If the above is satisfied, the first stable point disappears:



With enough current, the network loses its bi-stability, and neurons always engage in repeated firing.

Exercise 4

a) $\alpha(t) = \sigma(t) + \sigma(t - t_0)$

$$\sigma(t) = \begin{cases} b\, e^{-t/\tau_m} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

When a neuron fires, it elicits

$$\int_0^\infty \alpha(t)\, dt = 2 \int_0^\infty \sigma(t)\, dt = 2 b \tau_m$$

And because K neurons fire at rate ν_0 we have

$$\langle v \rangle = 2 K b \tau_m \nu_0$$

b1)

$$\langle v \rangle = 2 \cdot \cancel{1000} \cdot 1\,mV \cdot 10\,m\cancel{s} \cdot \cancel{10^{-3}}\, \cancel{ms^{-1}}$$

$$= 2\,mV$$

b2) No. Each individual neuron hovers at a potential much lower than required for spiking. (Neurons might still occasionally fire due to statistical fluctuations, although that is very unlikely for $K \to \infty$).

c) the mean potential would stay unchanged.
(It follows immediately from (d))

d) Standard deviation would decrease.
When $t_0 \gg t_m$, individual elements of the doublet are essentially like independent, individual spikes, canceling out more.
When $t_0 \to 0$, the two spikes are completely correlated, amplifying the fluctuations.

This result could be derived formally using the Law of total Variance and a bit of patience ☺.

Exercise 5

a) $\tau_I \frac{dh_i}{dt} = -h_i + \tau w_{IE} R + \tau w_{II} A(t)$

b) Assuming $h_{17}(0) = h_{366}(0)$, yes, the neurons will have the same value for the membrane potential.

Otherwise, $h_{17}(t)$ still converges to $h_{366}(t)$ for $t \to \infty$. To show this we would need to solve (a) using the usual technique we saw in the exercises of Week 1.

c) Assuming $\gamma \to \infty$, $h > 0$ we have $A = f(h) = \frac{1}{\tau} h$

$$\frac{dA}{dt} = \frac{d}{dt} f(h(t)) = \frac{d}{dt} \frac{h(t)}{\tau} = \frac{1}{\tau} \frac{dh}{dt}$$

Hence

$$\frac{dA}{dt} = \frac{1}{\tau \tau_I} \left(-\tau A + \tau w_{IE} R + \tau w_{II} A \right) =$$

$$= \frac{-A(1 - w_{II}) + w_{IE} R}{\tau_I}$$

d) From (c) we got

$$t_I \frac{dA}{dt} = -A(1-w_{II}) + w_{IE} r$$

$$\frac{t_I}{1-w_{II}} \frac{dA}{dt} = -A + \frac{w_{IE} r}{(1-w_{II})}$$

So A converges exponentially to $\frac{w_{IE} r}{(1-w_{II})}$

Assuming $t_I << t_E$ we can approximate

$$A(t) = \frac{w_{IE} r}{1-w_{II}}$$

So we have

$$t_E \frac{d}{dt} h_E = -h_E + RI_0 + t_E w_{EE} g(h_E) + t_E w_{EI} A(t)$$

$$= -h_E + RI_0 + t_E w_{EE} g(h_E) + t_E w_{EI} \frac{w_{IE} g(h_E)}{1-w_{II}}$$

Hence

d1) $\beta = \left(w_{EE} + \frac{w_{EI} w_{IE}}{1-w_{II}}\right)$ d2) $h_0 = RI_0$